Adding a Level-1 Predictor

PSYC 575

August 25, 2020 (updated: 9 September 2021)

Week Learning Objectives

- Explain what the **ecological fallacy** is
- Use **cluster-mean/group-mean centering** to **decompose** the effect of a lv-1 predictor
- Define **contextual effects**
- Explain the concept of **random slopes**
- Analyze and interpret **cross-level interaction** effects

Adding Level-1 Predictors

- E.g., student's SES
- Both predictor (`ses`) and outcome (`mathach`) are at level 1
- OLS still has Type I error inflation problem
  - Unless ICC = 0 for the predictor
- MLM can answer additional research questions
  - Within-Between effects and contextual effects
  - Random (varying) slopes
  - Cross-level interactions

Research Questions

- Does math achievement vary across schools? How much is the variation?
- Do schools with higher mean SES have students with higher math achievement?
- Do students with higher SES have higher math achievement? Is the relation similar at the individual and cluster levels? Is this relation similar across schools?
- Is the relation between SES and math achievement moderated by some types of schools (e.g., Catholic vs. Public, high mean SES vs low mean SES)?

The Same Predictor?

- Is it different to use MEANSES vs. SES as predictor?
  - MEANSES → MATHACH is positive
  - γ_{01} = 5.72 (*SE* = 0.18)
- Should the coefficient be the same with SES?

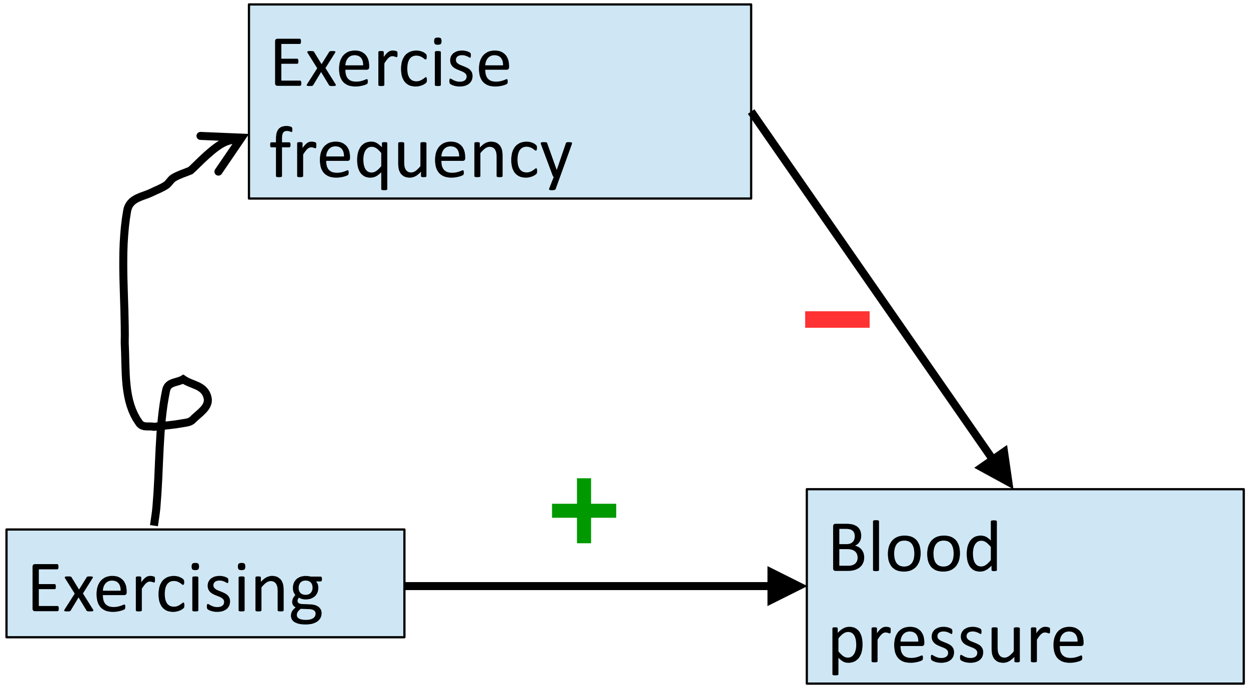
Ecological Fallacy

Ecological Fallacy

- Robinson’s paradox (% immigrant and % illiterate)
- Errors in assuming that relationships at one level are the same moving to another level
- Failure to account for the clustering structure
  ➔ <u>Misleading results</u>

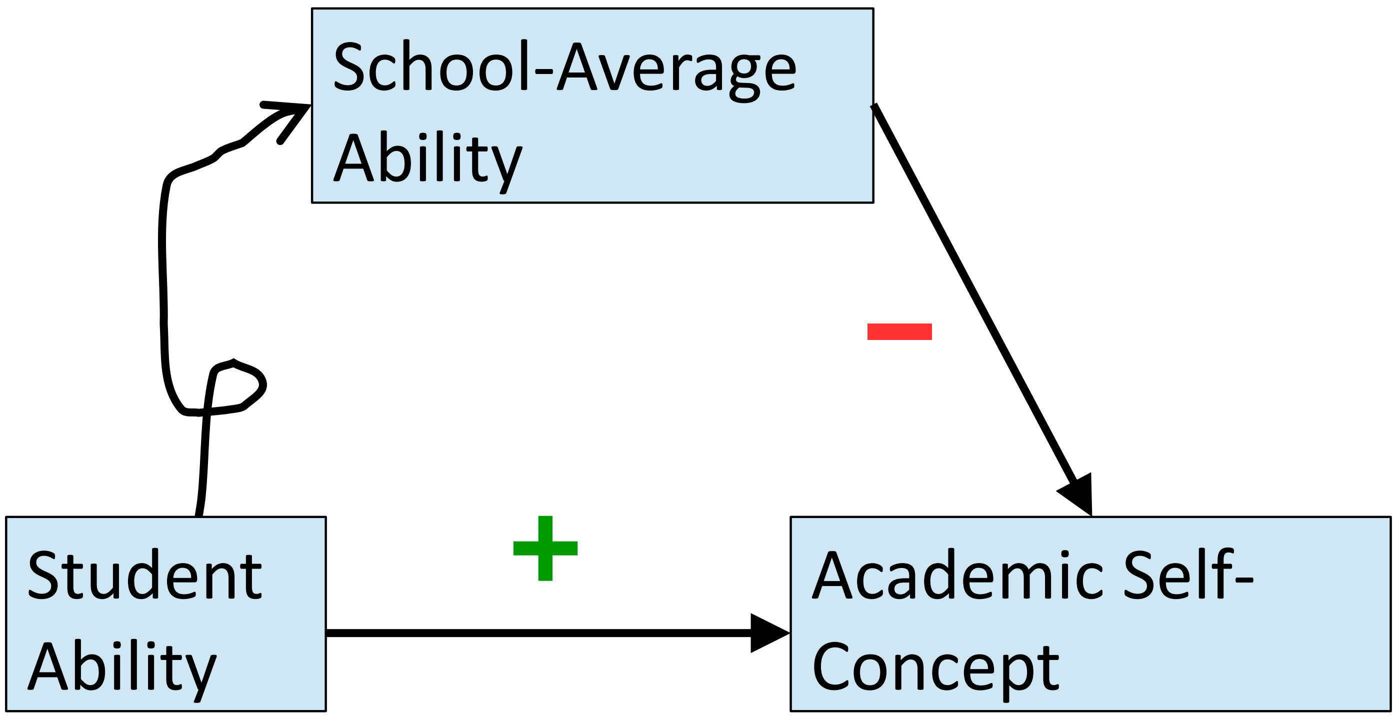
“Same” Predictor, Different Effects

- Example: Exercise and blood pressure

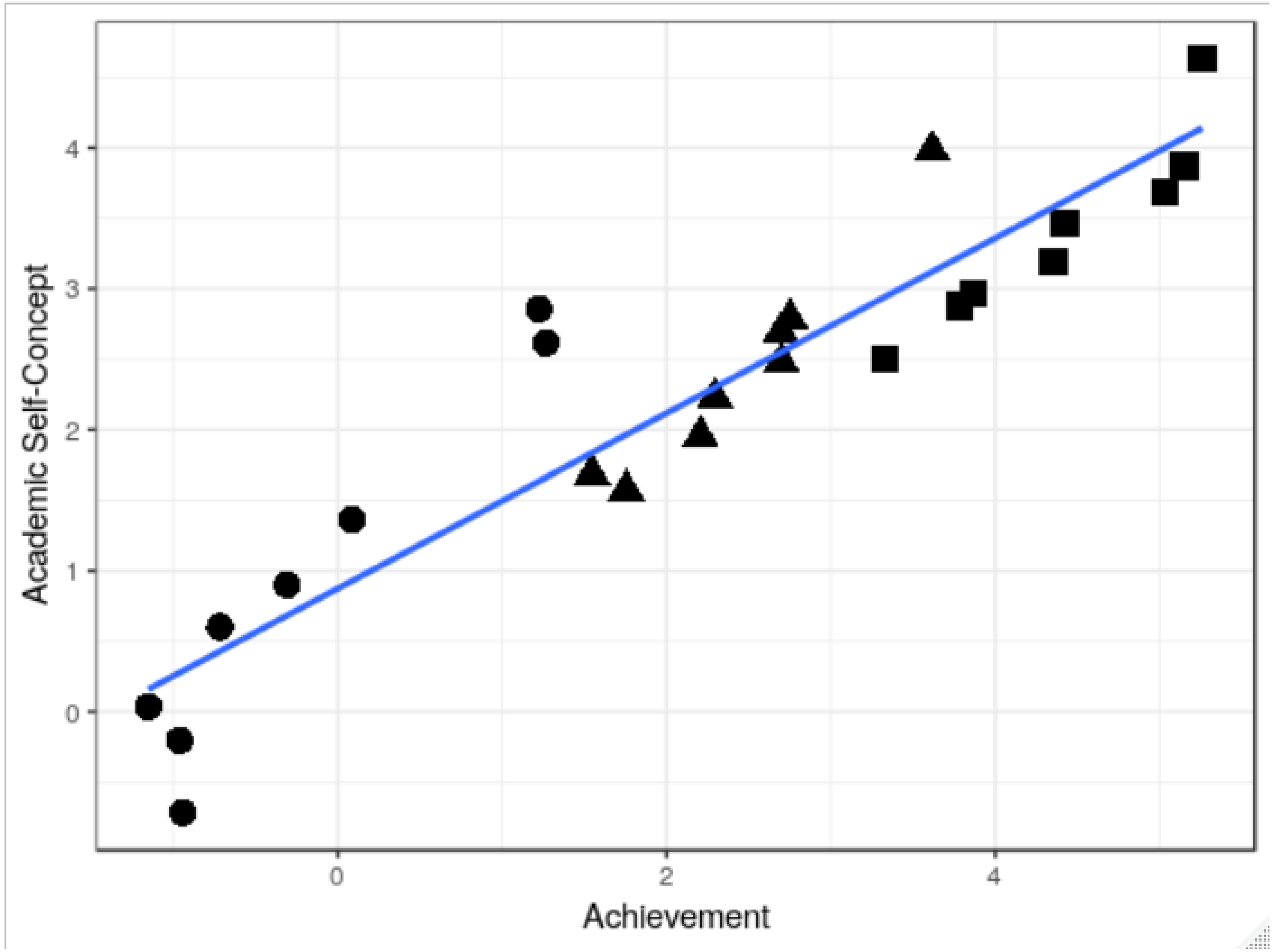


“Same” Predictor, Different Effects

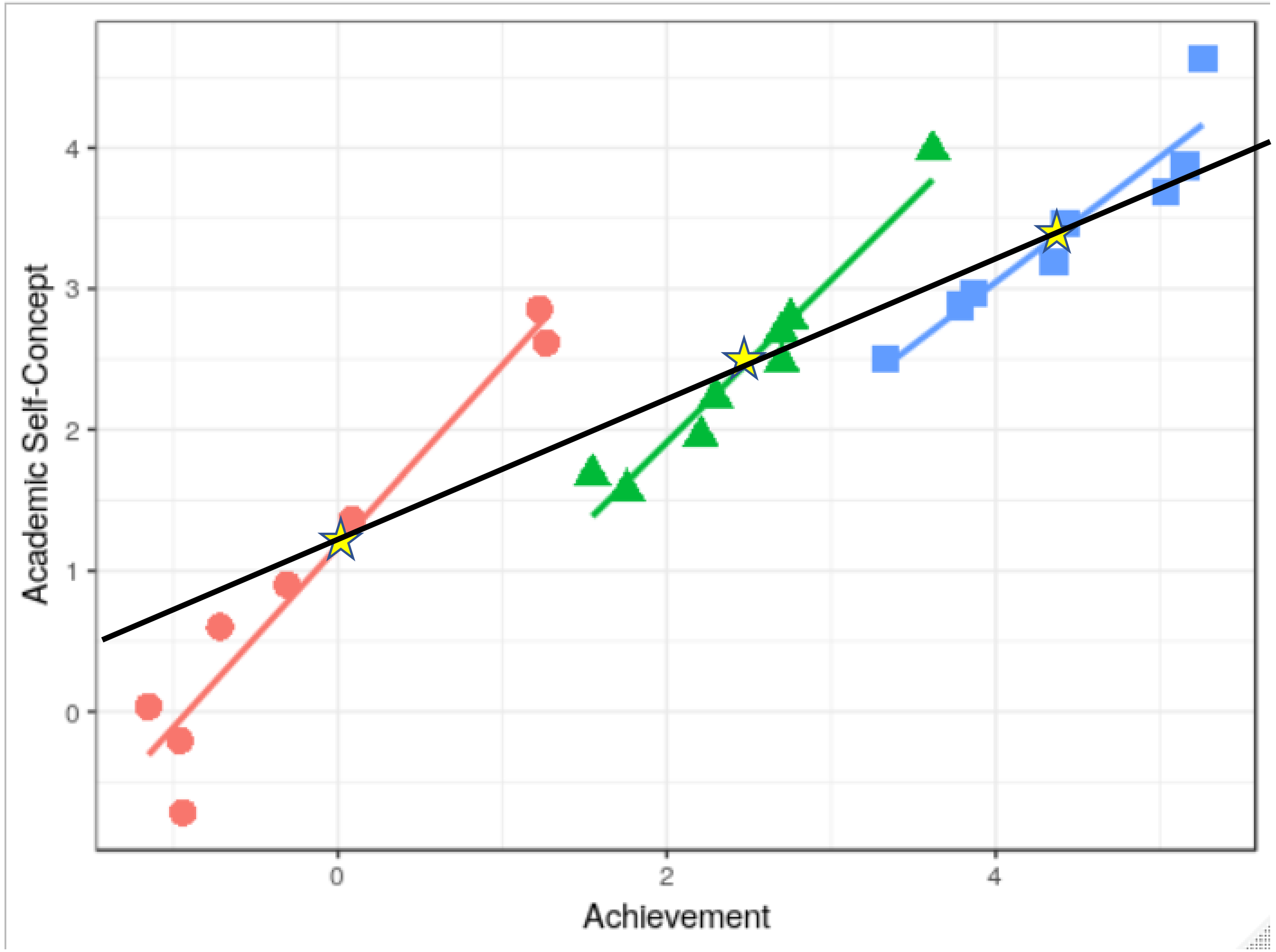
- Example: Big-Fish-Little-Pond Effect (Marsh & Parker, 1984)



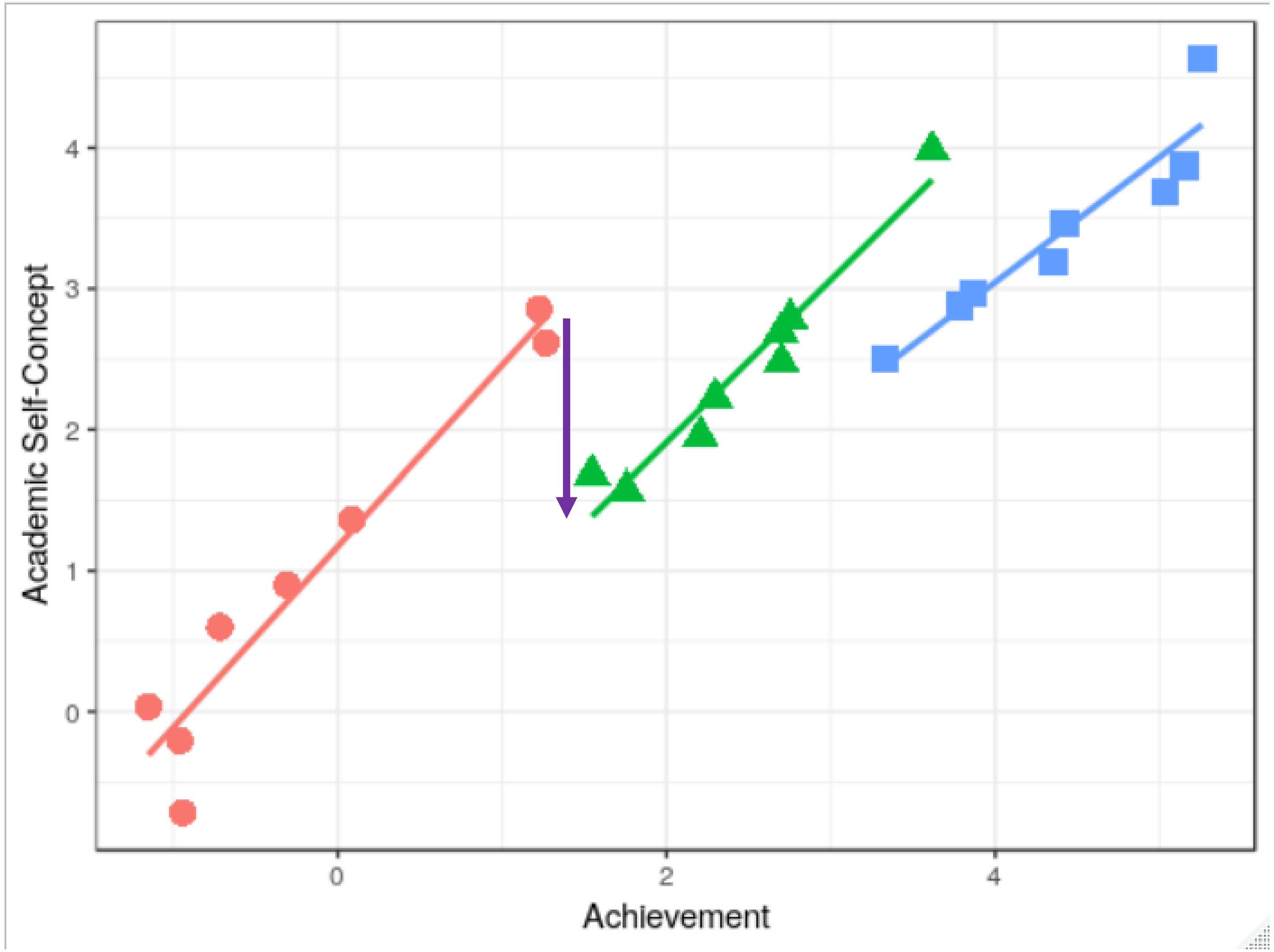
Overall Effect



Within & Between Effects



Within & Contextual Effects



Never simply include a level-1 predictor

Unless it has the same values for every cluster

Additional reference: Antonakis, Bastardoz & Ronkko (2021, https://doi.org/10.1177/1094428119877457)

Two Approaches

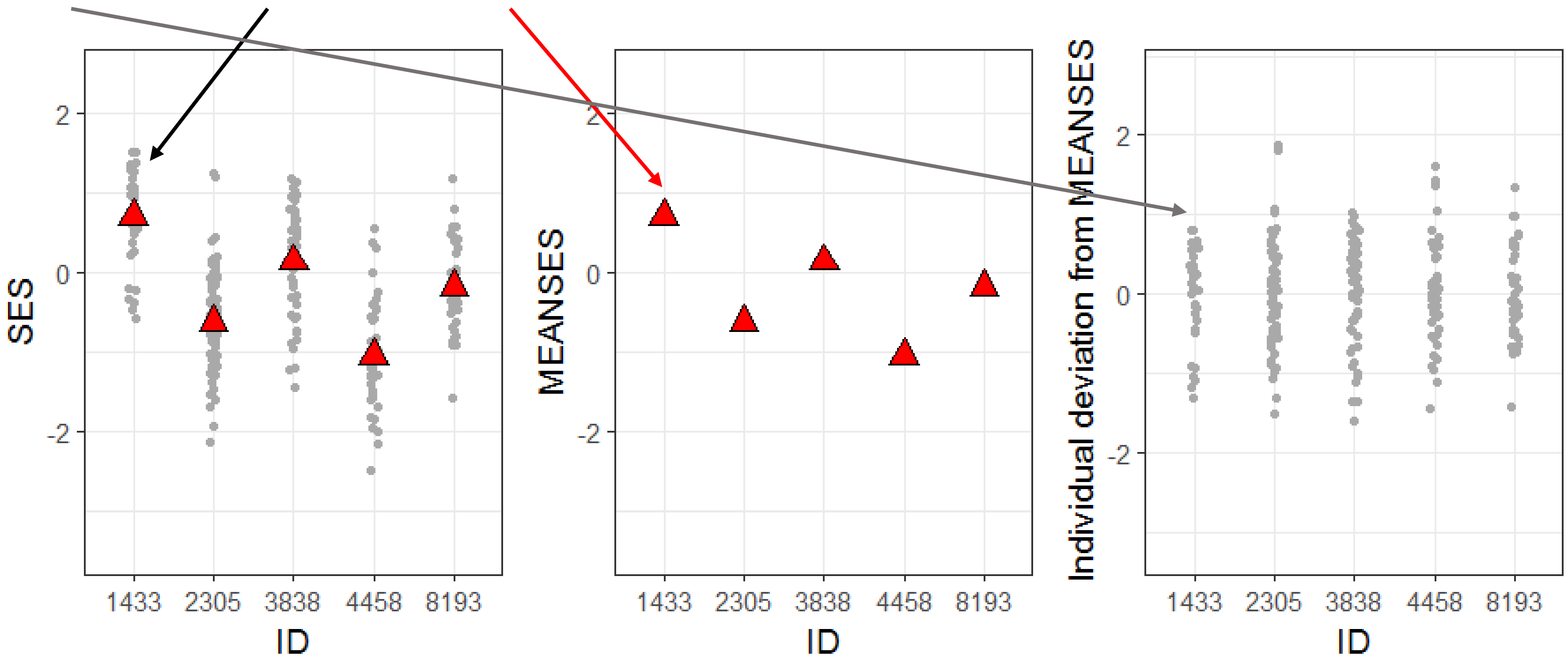
- Both involves computing the cluster means
  - E.g., `ses` → `meanses`

1. Cluster-mean centered (cmc) variable + cluster mean
   - Between-within method
   - Decompose into between-within effects
2. Raw/uncentered predictor + cluster mean
   - Study contextual effects (i.e., between minus within)

`mathach` vs. `ses`

Decomposing Into Lv-2 and Lv-1 Components

- Group-mean centering
  - `ses_cmc` = $\text{ses}_{ij} - \text{meanses}_j$

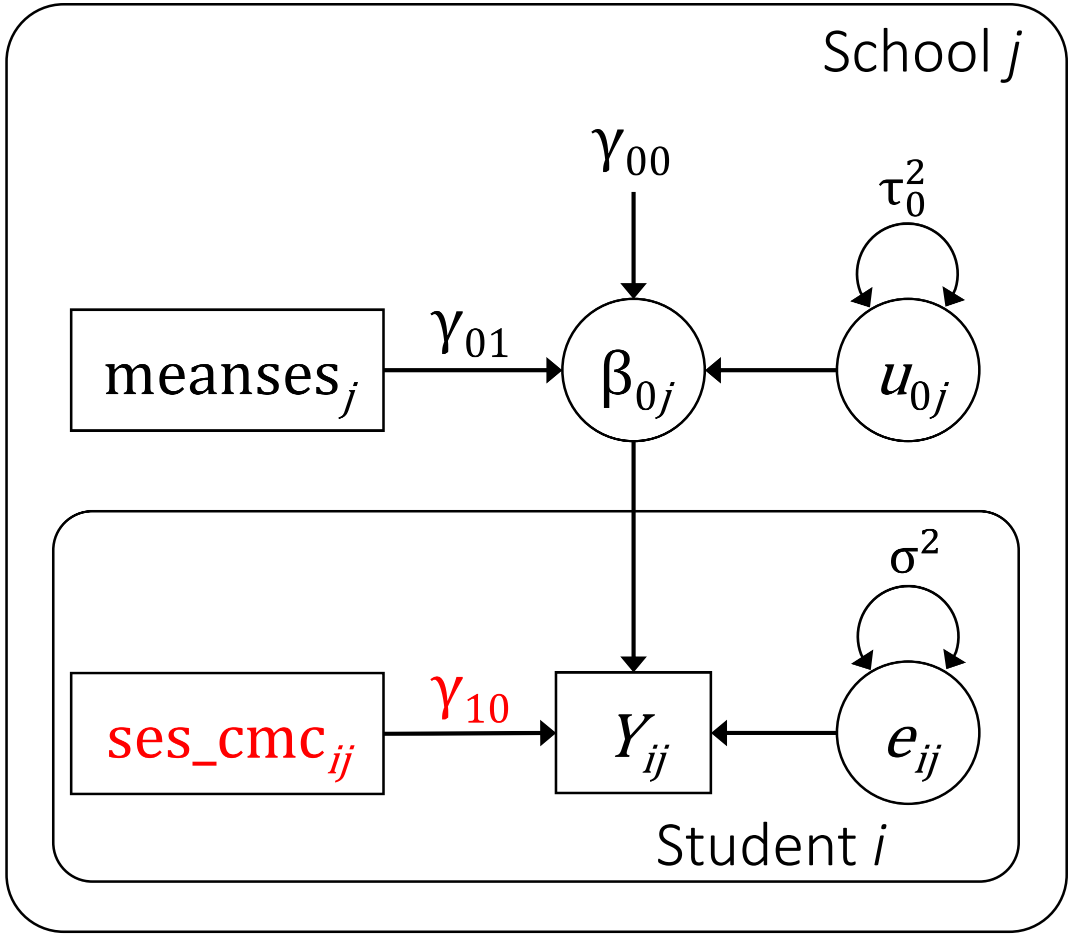


Between-Within Decomposition

- Lv 1:
  $\text{mathach}_{ij} = \beta_{0j} + \beta_{1j}\, \text{ses\_cmc}_{ij} + e_{ij}$
- Lv 2:
  $\beta_{0j} = \gamma_{00} + \gamma_{01}\, \text{meanses}_j + u_{0j}$
  $\beta_{1j} = \gamma_{10}$
- Combined:
  $\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\, \text{meanses}_j + \gamma_{10}\, \text{ses\_cmc}_{ij} + u_{0j} + e_{ij}$

Student-level Effect

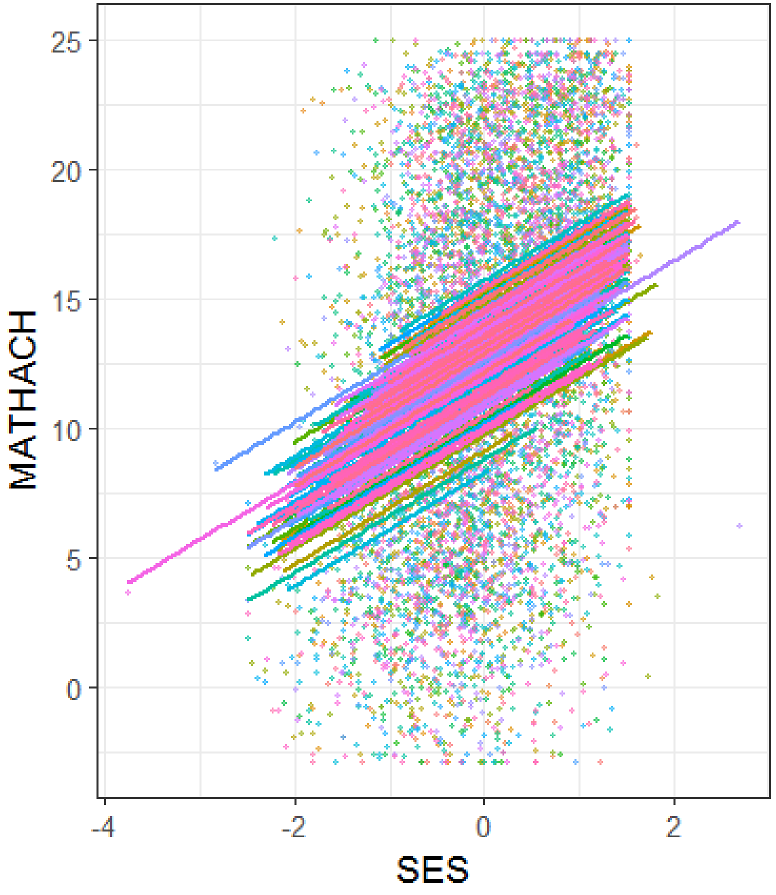
School-level Effect

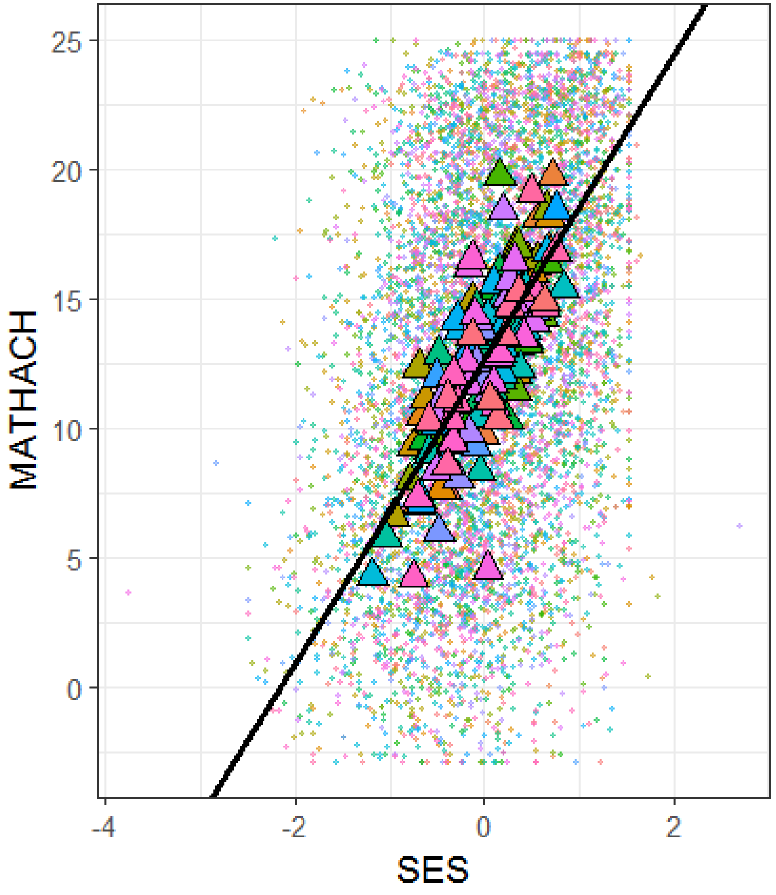


```
># Linear mixed model fit by REML ['lmerMod']
># Formula: mathach ~ meanses + ses_cmc + (1 | id)
>#    Data: hsball

># Fixed effects:
>#             Estimate Std. Error t value
># (Intercept)  12.6481     0.1494   84.68
># meanses       5.8662     0.3617   16.22
># ses_cmc       2.1912     0.1087   20.16
```

The student-level effect is 2.19
The school-level effect is 5.87

Visualizing the Difference





Interpret the Coefficients

- Student A
  - From a school of average SES
  - SES level at the school mean

  ➔ `ses` = ____, `meanses` = ___, `ses_cmc` = ___
- Predicted `mathach` = ___ + ___ (___) + ___ (___)

  = ___

Interpret the Coefficients

- Student B
  - From a school of average SES
  - SES level 1 unit higher than the school mean

  ➔ `meanses` = ___, `ses_cmc` = ___
- Predicted `mathach` = ___ + ___ (___) + ___ (___)

  = ___

Interpret the Coefficients (Cont'd)

- Student C
  - From a high SES school (one unit higher than average)
  - SES level 1 unit below the school mean

  ➔ `meanses` = ___, `ses_cmc` = ___

- Predicted `mathach` = ___ + ___ (___) + ___ (___)

  = ___

Contextual Effects

Contextual Effect[1]

- γ_{01} - γ_{10} = 5.87 – 2.19 = 3.68
- Effect of School SES (context) on individuals:
  - Expected difference in achievement between two students with same SES, but from schools with a 1 unit difference in `meanses`

[1]: When there is no random slopes, the contextual effect model is a reparameterization of the between-within model, meaning that they have the same fit

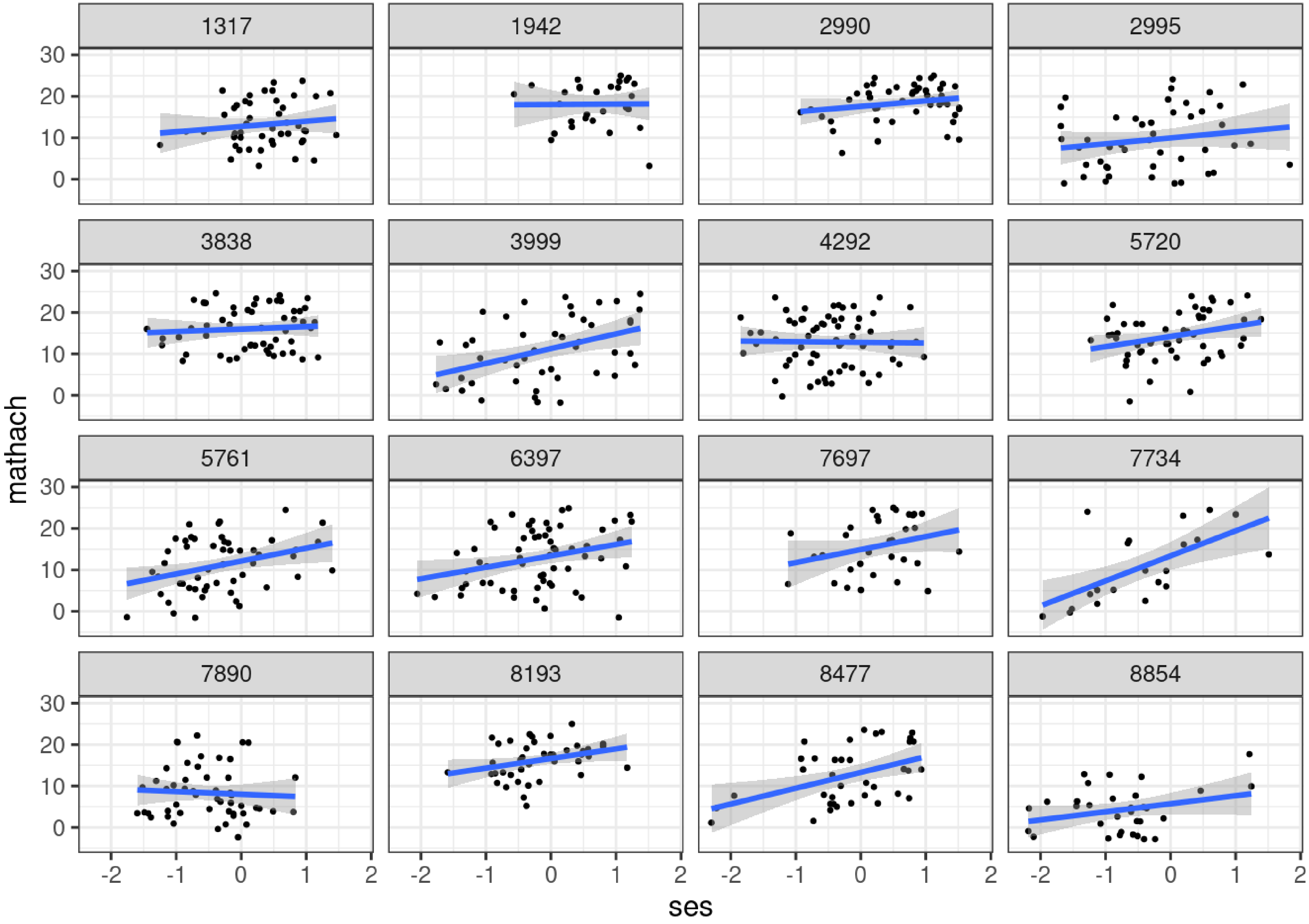
```
># Linear mixed model fit by REML ['lmerMod']
># Formula: mathach ~ meanses + ses + (1 | id)
>#    Data: hsball

># Fixed effects:
>#             Estimate Std. Error t value
># (Intercept)  12.6613     0.1494  84.763
># meanses       3.6750     0.3777   9.731
># ses           2.1912     0.1087  20.164
```

The student-level effect is 2.19;
the contextual effect
= 3.68 = 5.87 – 2.19

Random Slopes/Random Coefficients

Research Questions

- Does math achievement varies across schools? How much is the variation?
- Do schools with higher mean SES have students with higher math achievement?
- Do students with higher SES have higher math achievement? Is the relation similar at the individual and cluster levels? Is this relation similar across schools?
- Is the relation between SES and math achievement moderated by some types of schools (e.g., Catholic vs. Public, high mean SES vs low mean SES)?



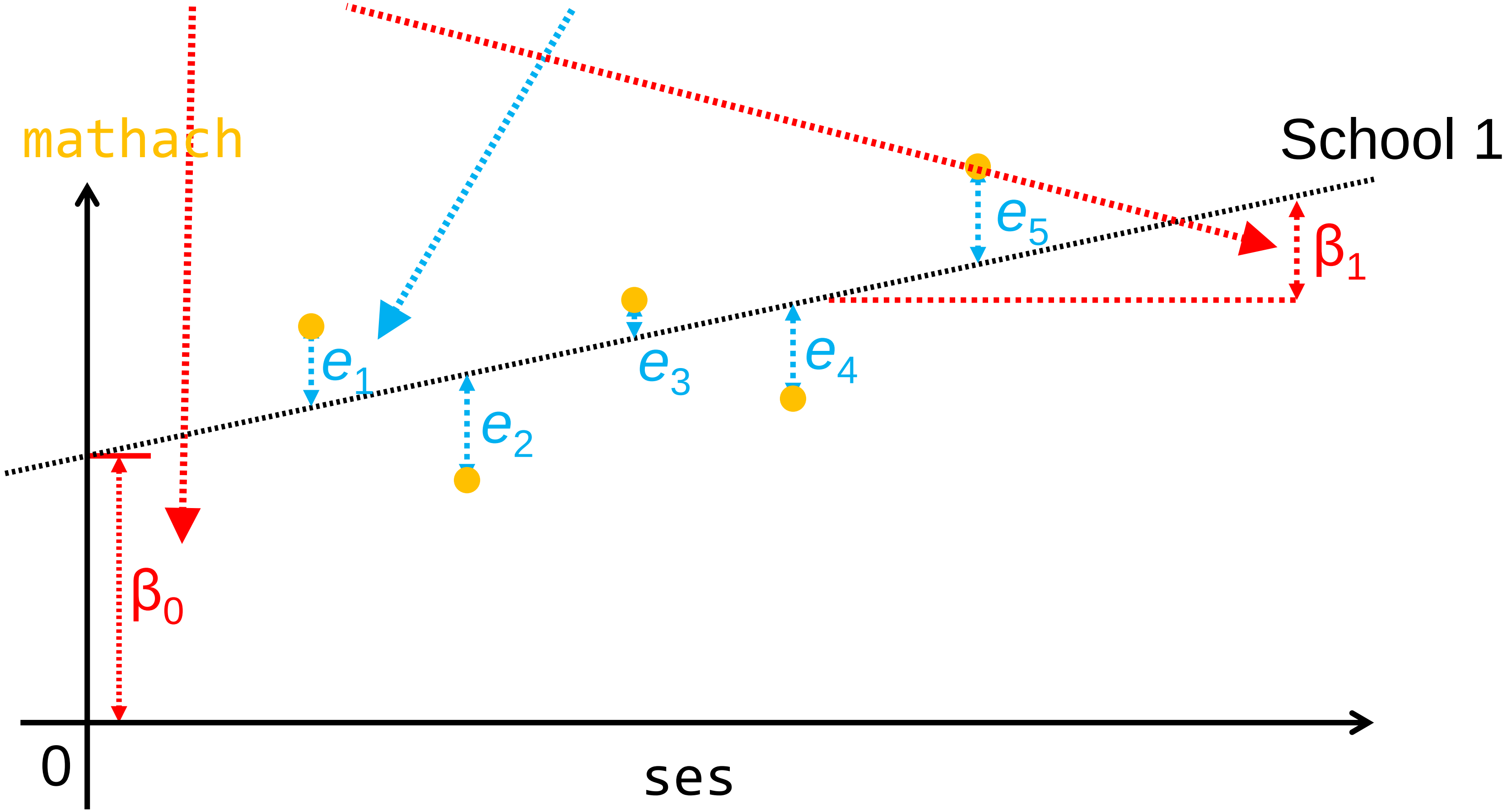
1317
1942
2990
2995
3838
3999
4292
5720
5761
6397
7697
7734
7890
8193
8477
8854
mathach
ses
30
20
10
0
-2
-1
0
1
2

Varying Regression Lines

- Decomposing effect model
  - Assumes constant slope across schools for `ses` → `mathach`
- Instead, one can investigate whether that relation changes across schools

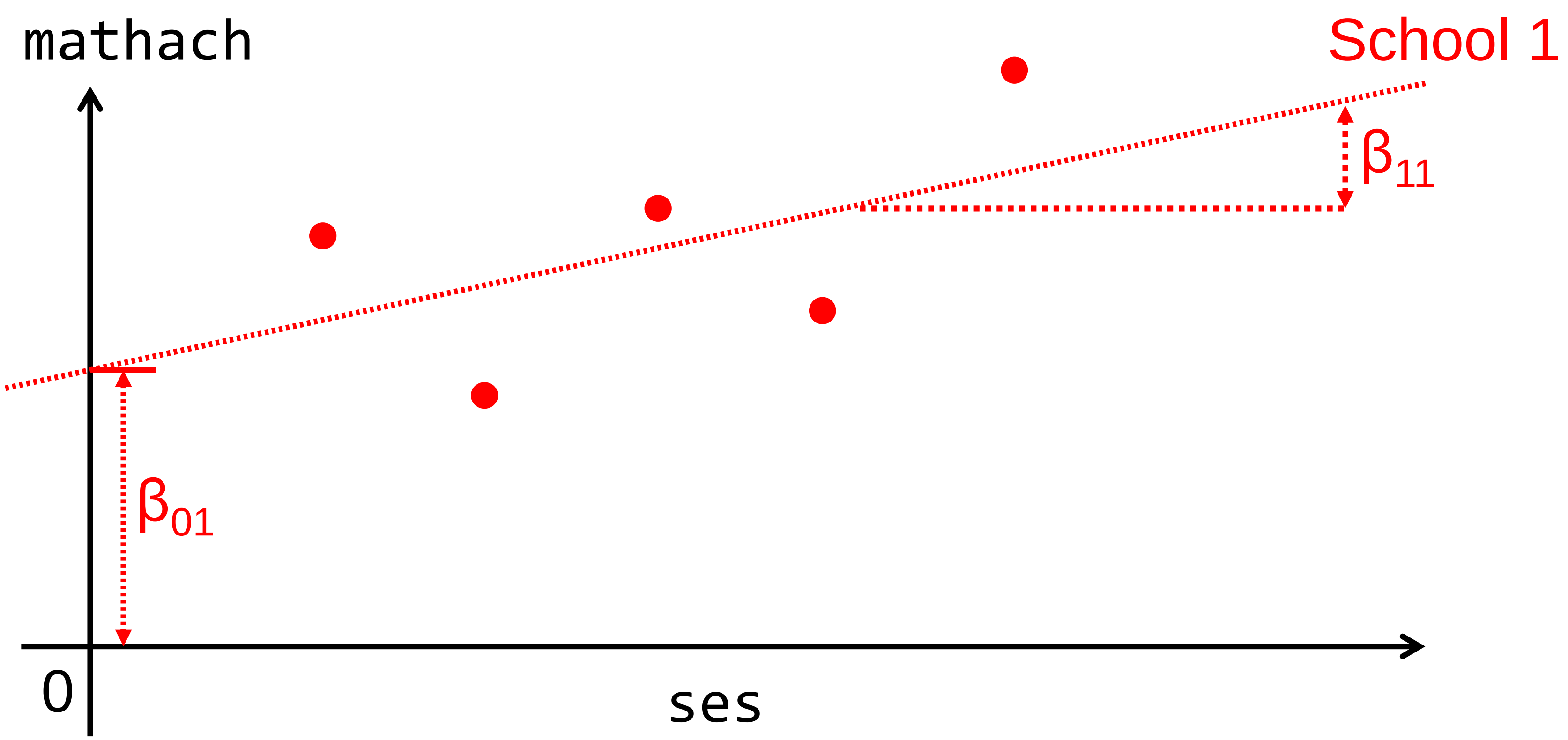
Let's Focus on One School

- $\text{mathach}_i = \beta_0 + \beta_1 \text{ses}_i + e_i$



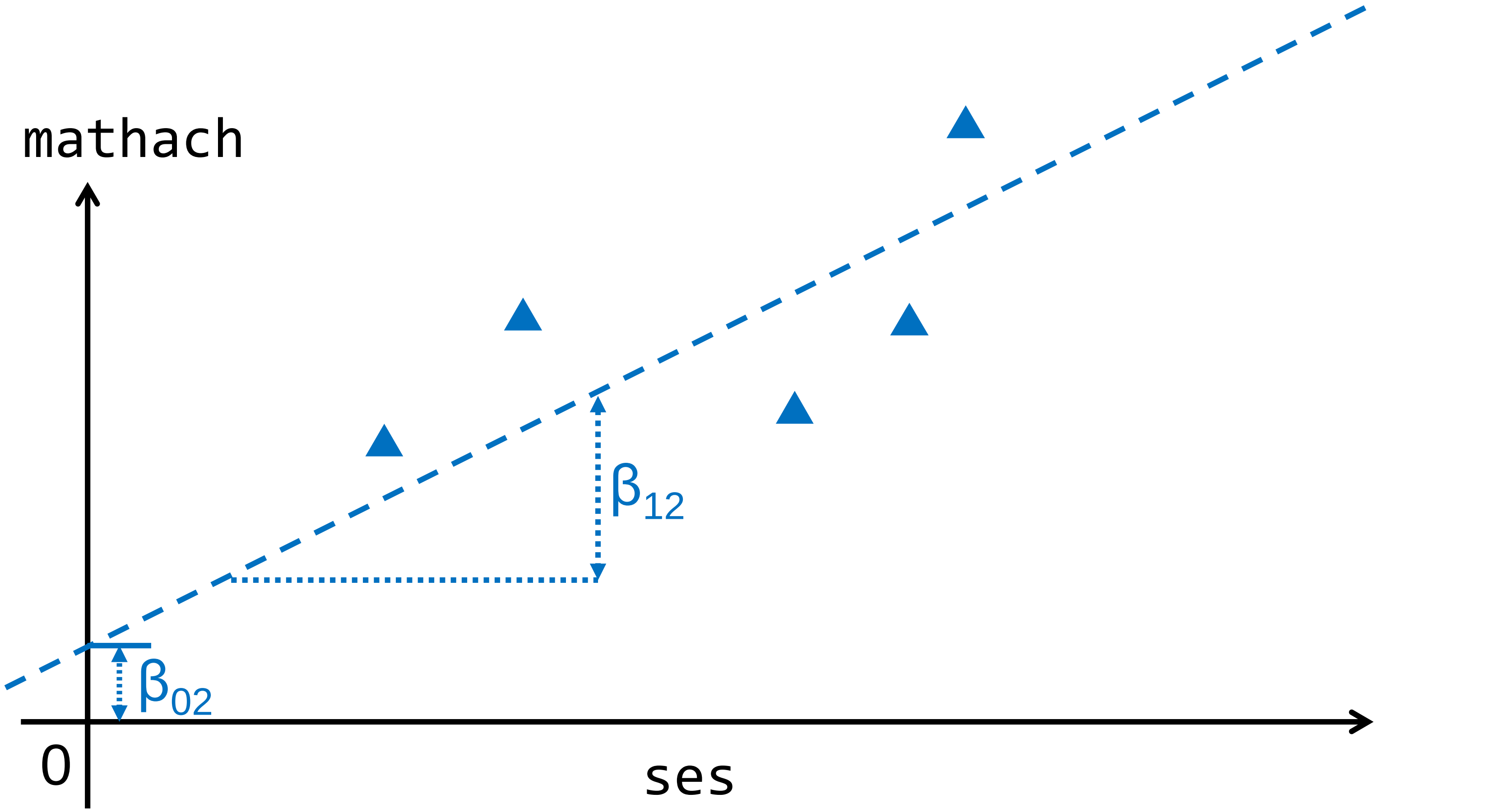
Multi-Level Model (MLM)

- School 1: $\text{mathach}_{i1} = \beta_{01} + \beta_{11}\text{ses}_{i1} + e_{i1}$



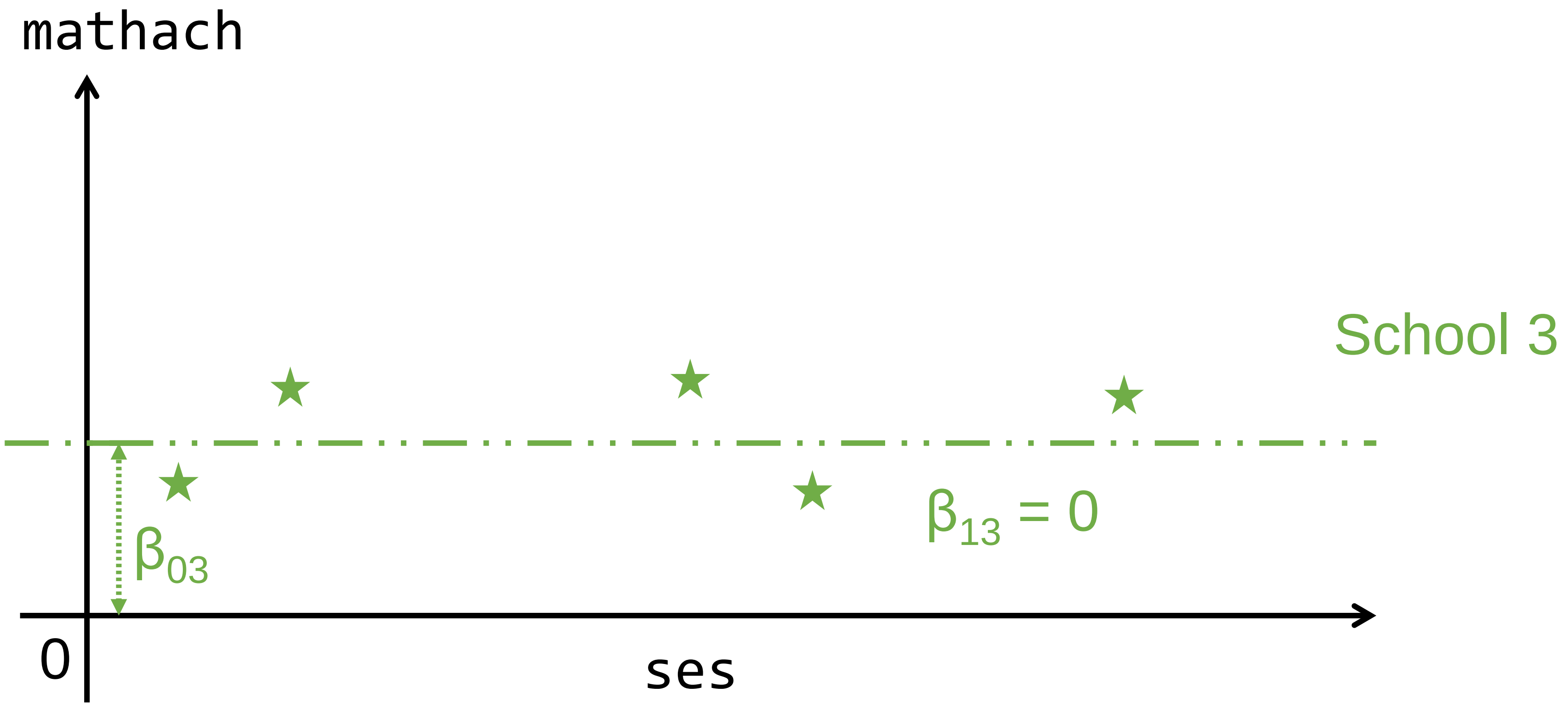
Consider a Second School

- School 2: $\text{mathach}_{i2} = \beta_{02} + \beta_{12}\text{ses}_{i2} + e_{i2}$

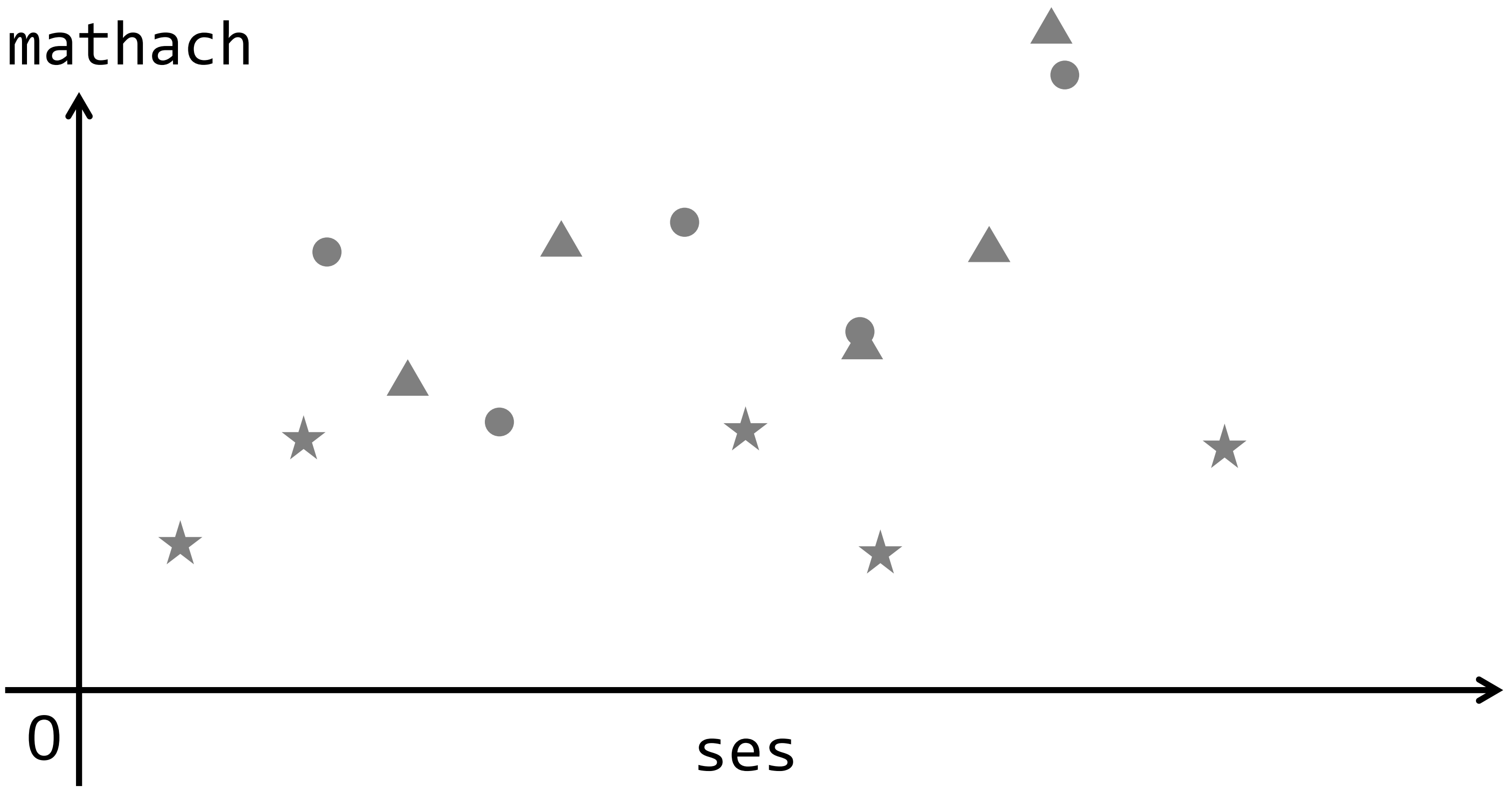


Consider a Third School

- School 3: $\text{mathach}_{i3} = \beta_{03} + \beta_{13}\text{ses}_{i3} + e_{i3}$

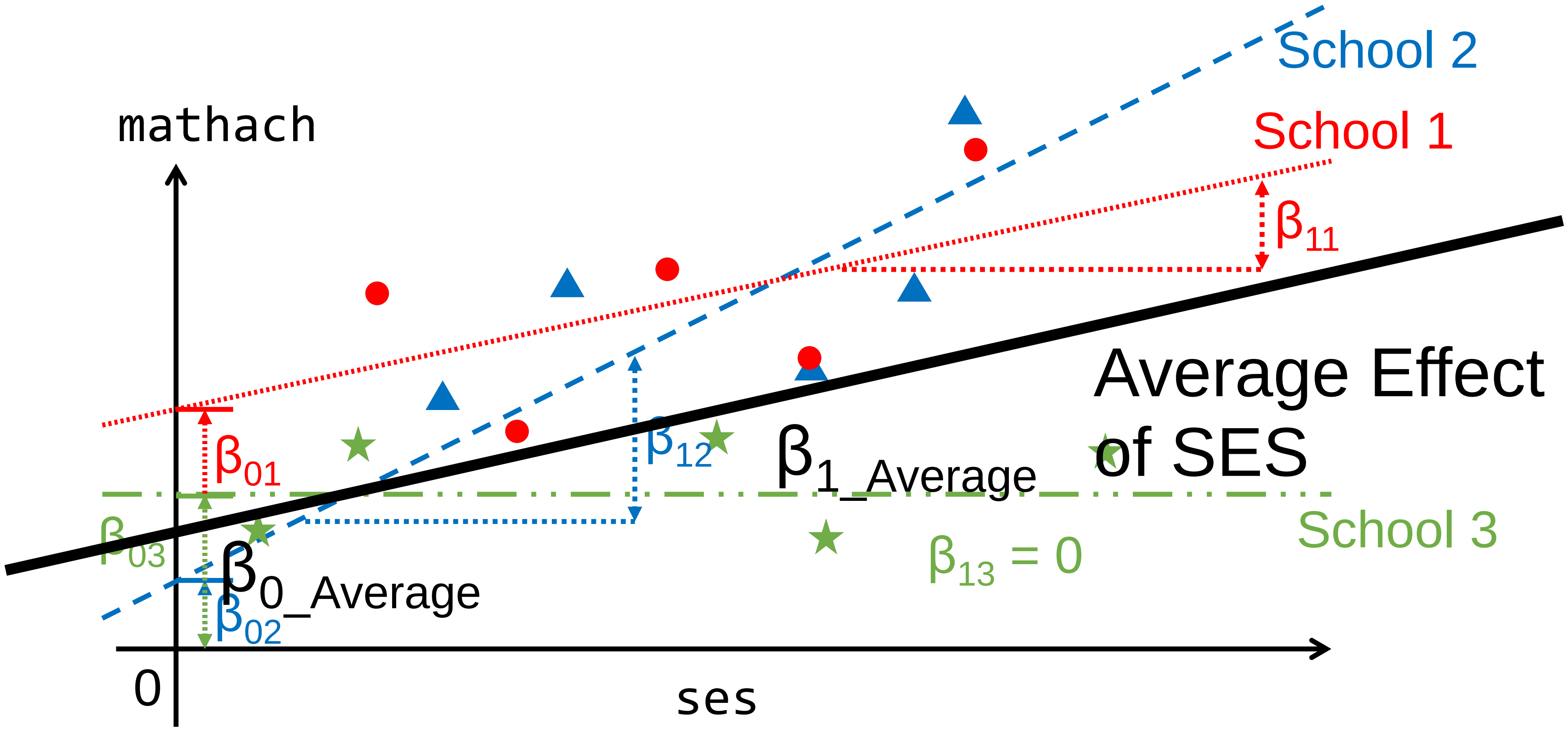


Combining All Schools



Combining All Schools

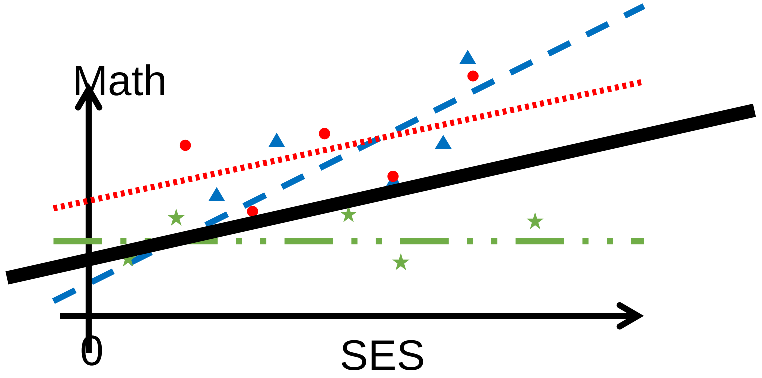
- $\text{mathach}_{ij} = \beta_{0j} + \beta_{1j}\text{ses}_{ij} + e_{ij}$ (j = 1, 2, … , 160)



Combining All Schools

School	β_{0j}	β_{1j}
1224	11.06	2.50
1288	13.07	2.48
1296	9.20	2.35
1308	14.38	2.31
…		
9397	10.40	1.87
9508	13.69	2.52
9550	11.29	2.67
9586	13.37	2.27
Mean	13.01	2.39
Variance	4.83	0.41

160 Schools



Random Intercepts

Random Slopes

$\beta_{1\_Average} = \gamma_{10}$

$\beta_{0\_Average} = \gamma_{00}$

Random-Coefficient Model

- Lv 1:
  - $\text{mathach}_{ij} = \beta_{0j} + \beta_{1j}\,\text{ses\_cmc}_{ij} + e_{ij}$
- Lv 2:
  - $\beta_{0j} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + u_{0j}$
  - $\beta_{1j} = \gamma_{10} + u_{1j}$
- Combined:
  - $\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + \gamma_{10}\,\text{ses\_cmc}_{ij} + u_{0j} + u_{1j}\,\text{ses\_cmc}_{ij} + e_{ij}$

Average slope of SES

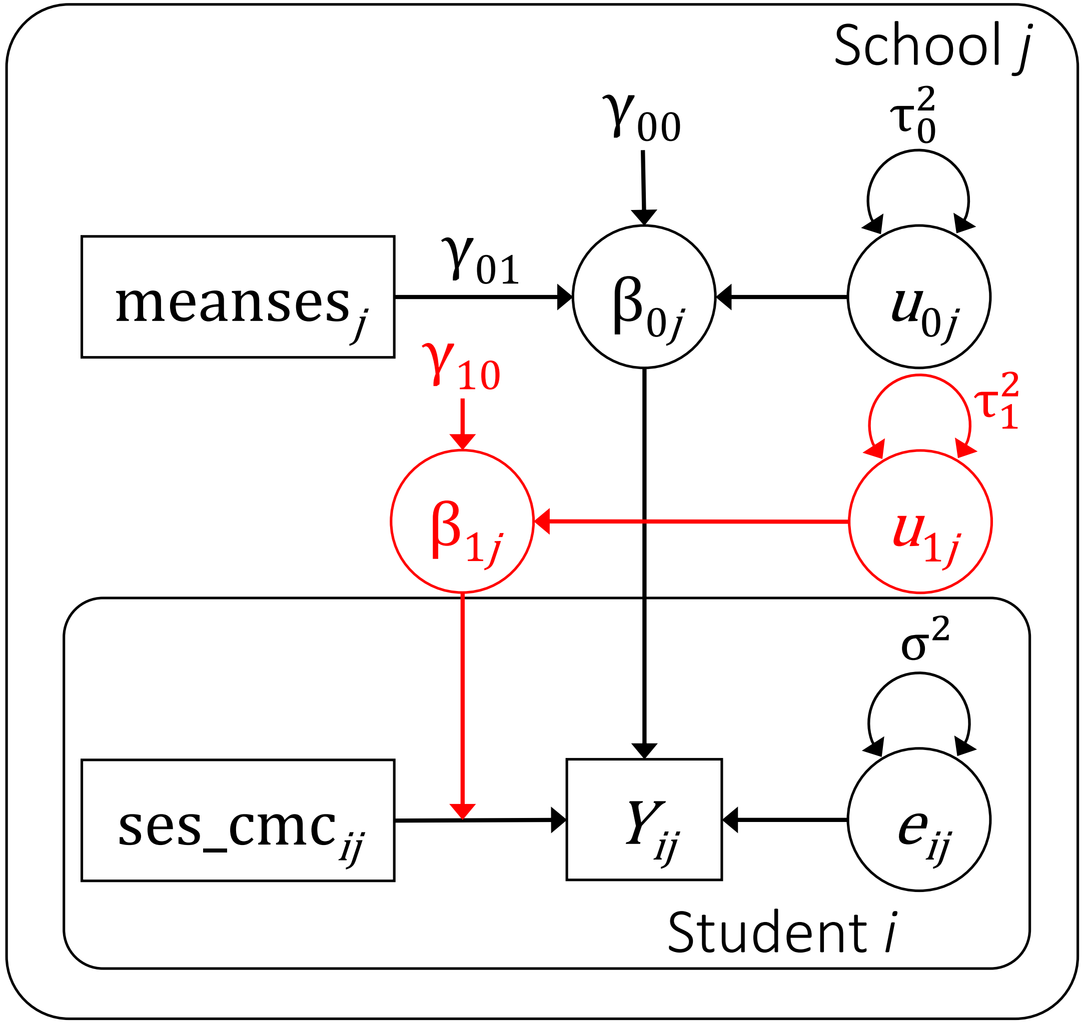
Deviation of school *j*'s slope from the average

Centering

- Raudenbush & Bryk (2002) noted that slope variance were better estimated with cluster mean centering
  - However, Snijders & Bosker (5.3.1) suggested it should be based on theory
- Remember to add the cluster means
- See also consult Enders & Tofighi (2007)[1]

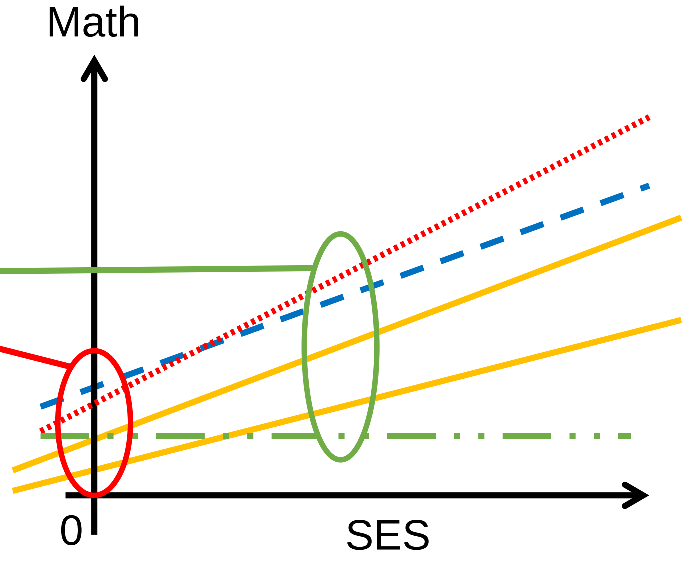
[1]: https://doi.org/10.1037/1082-989X.12.2.121

Path Diagram



Variance Components

- $\mathrm{Var}(u_{0j}) = \tau_0^2$
- $\mathrm{Var}(u_{1j}) = \tau_1^2$



$$\mathrm{Var}\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} = \mathbf{G} = \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{bmatrix}$$

Variance of the school intercepts

Covariance of the intercepts and slopes, which are ***seldom interpreted***

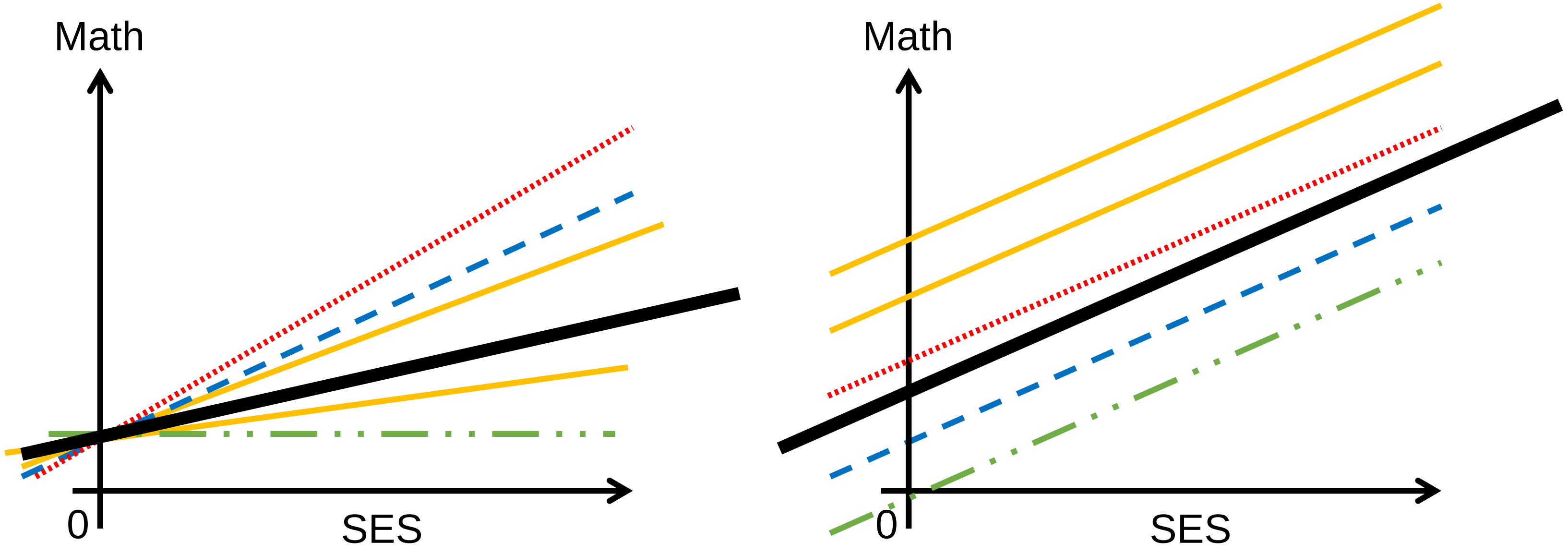
Variance of the school slopes

No random intercepts

$\mathrm{Var}(u_{0j}) = \tau_0^2 = 0$

No random slopes

$\mathrm{Var}(u_{1j}) = \tau_1^2 = 0$



Full Equations

$$\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\text{meanses}_j + \gamma_{10}\text{ses\_cmc}_{ij} + u_{0j} + u_{1j}\text{ses\_cmc}_{ij} + e_{ij}$$

$$\binom{u_{0j}}{u_{1j}} \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\tau_0^2 & \tau_{01}\\ \tau_{01} & \tau_1^2\end{bmatrix}\right)$$

$$e_{ij} \sim N(0, \sigma)$$

Look at the *SE*s of Fixed Effects

```
> lmer(mathach ~ meanses + ses_cmc + (ses_cmc | id), data = hsball)

Fixed effects:
            Estimate Std. Error t value
(Intercept)  12.6454     0.1492   84.74
meanses       5.8963     0.3600   16.38
ses_cmc       2.1913     0.1280   17.12
```

SE = 0.109 when random slopes not included

➔ underestimated

Random Effect Estimates

```
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 id       (Intercept)  2.6931  1.6411
          ses_cmc      0.6858  0.8282   -0.19
 Residual             36.7132  6.0591
Number of obs: 7185, groups:  id, 160
```

- τ_0^2 = 2.69 = variance of intercepts
- τ_1^2 = 0.69 = slope variance

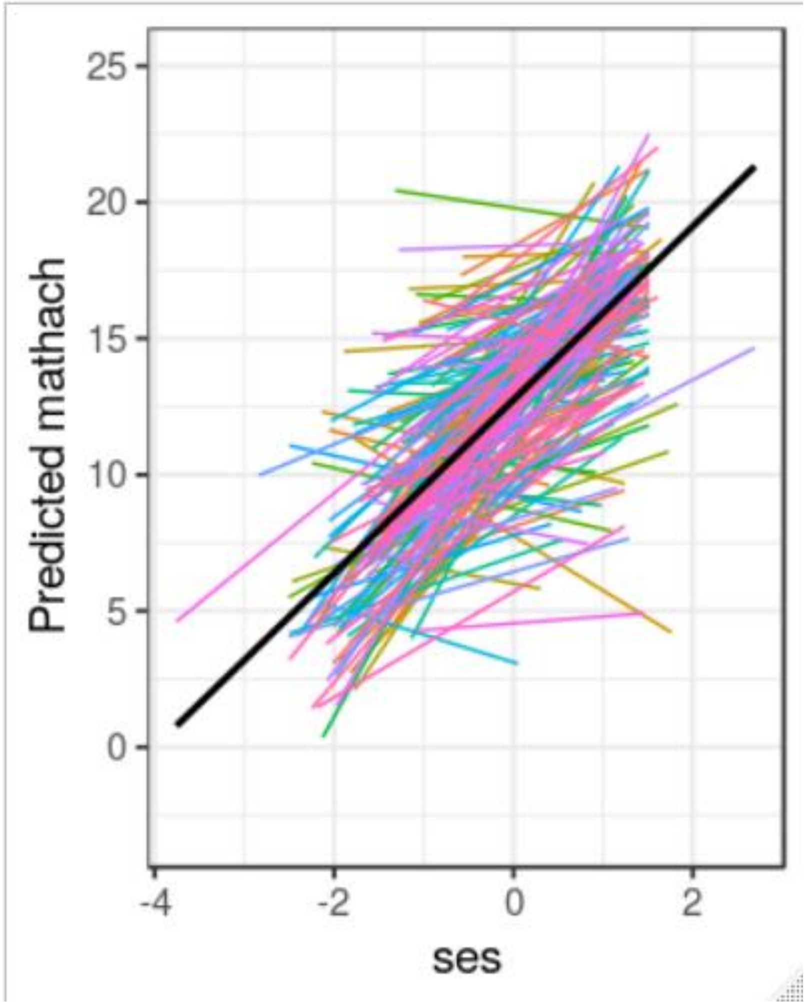
Interpreting Random Slopes

- Average slope = γ_{10} = 2.19
- *SD* of slopes = τ_1 = 0.83
- 68% Plausible range
  - γ_{10} +/- τ_1 = [$\gamma_{10} - \tau_1$, $\gamma_{10} + \tau_1$]
    = [_____, _____]

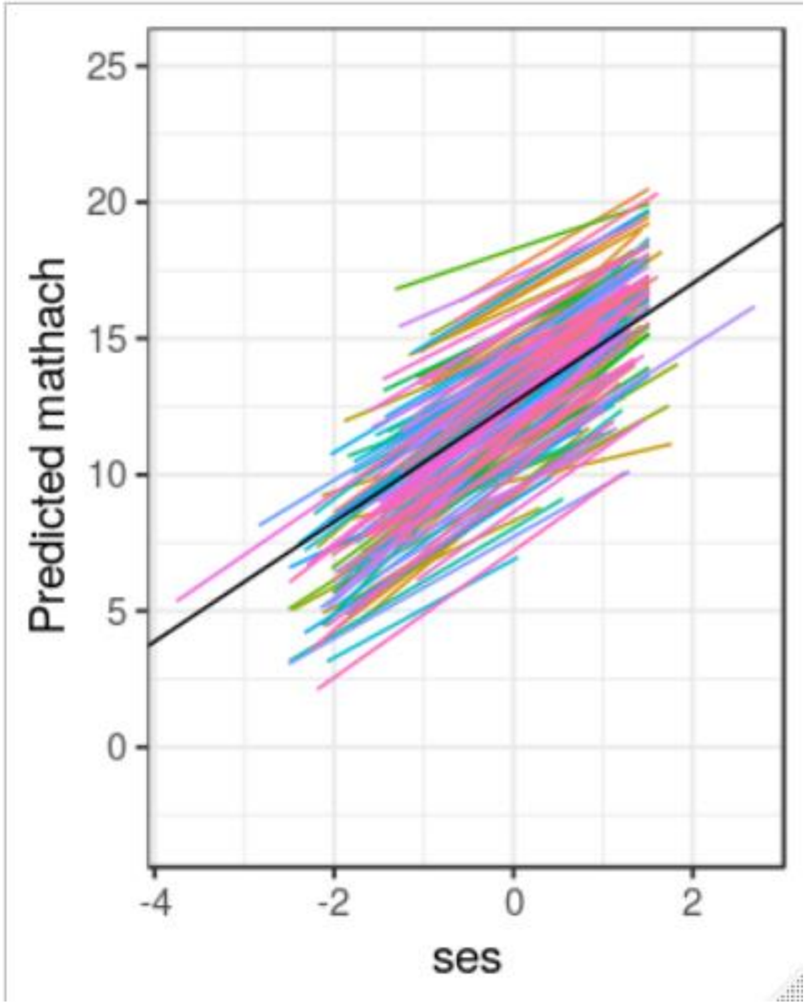
For majority of schools, SES and achievement are positively associated, with regression coefficients between ____ and _____

Visualize the Varying Slopes

OLS



Shrinkage (EB)



Cross-Level Interaction

Research Questions

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- Do schools with higher mean SES have students with higher math achievement?
- Do students with higher SES have higher math achievement? Is the relation similar at the individual and cluster levels? Is this relation similar across schools?
- Is the relation between SES and math achievement moderated by some types of schools (e.g., Catholic vs. Public, high mean SES vs low mean SES)?

Cross-Level Interaction

- Whether school-level variables <u>moderate</u> student-level relationships between variables
- Also called an <u>intercepts and slopes-as-outcomes model</u>
- Let’s add another school-level variable: `sector`
  - 1 = Catholic (*n* = 70), 0 = Public (*n* = 90)

Model Equations

- Lv 1:
  - $\text{mathach}_{ij} = \beta_{0j} + \beta_{1j}\,\text{ses\_cmc}_{ij} + e_{ij}$
- Lv 2:
  - $\beta_{0j} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + \gamma_{02}\,\text{sector}_j + u_{0j}$
  - $\beta_{1j} = \gamma_{10} + \gamma_{11}\,\text{sector}_j + u_{1j}$
- Combined:
  - $\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + \gamma_{10}\,\text{ses\_cmc}_{ij} + \gamma_{02}\,\text{sector}_j + \gamma_{11}\,\text{sector}_j \times \text{ses\_cmc}_{ij} + u_{0j} + u_{1j}\text{ses\_cmc}_{ij} + e_{ij}$

Main Effect of SECTOR

Cross-level product (interaction) term

Model Equations (cont'd)

- Lv 1:
  - $\text{mathach}_{ij} = \beta_{0j} + \beta_{1j}\,\text{ses\_cmc}_{ij} + e_{ij}$
- Lv 2:
  - $\beta_{0j} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + \gamma_{02}\,\text{sector}_j + u_{0j}$
  - $\beta_{1j} = \gamma_{10} + \gamma_{11}\,\text{sector}_j + u_{1j}$
- Combined:
  - $$\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\,\text{meanses}_j + \gamma_{10}\,\text{ses\_cmc}_{ij} + \gamma_{02}\,\text{sector}_j + \gamma_{11}\,\text{sector}_j \times \text{ses\_cmc}_{ij} + u_{0j} + u_{1j}\,\text{ses\_cmc}_{ij} + e_{ij}$$

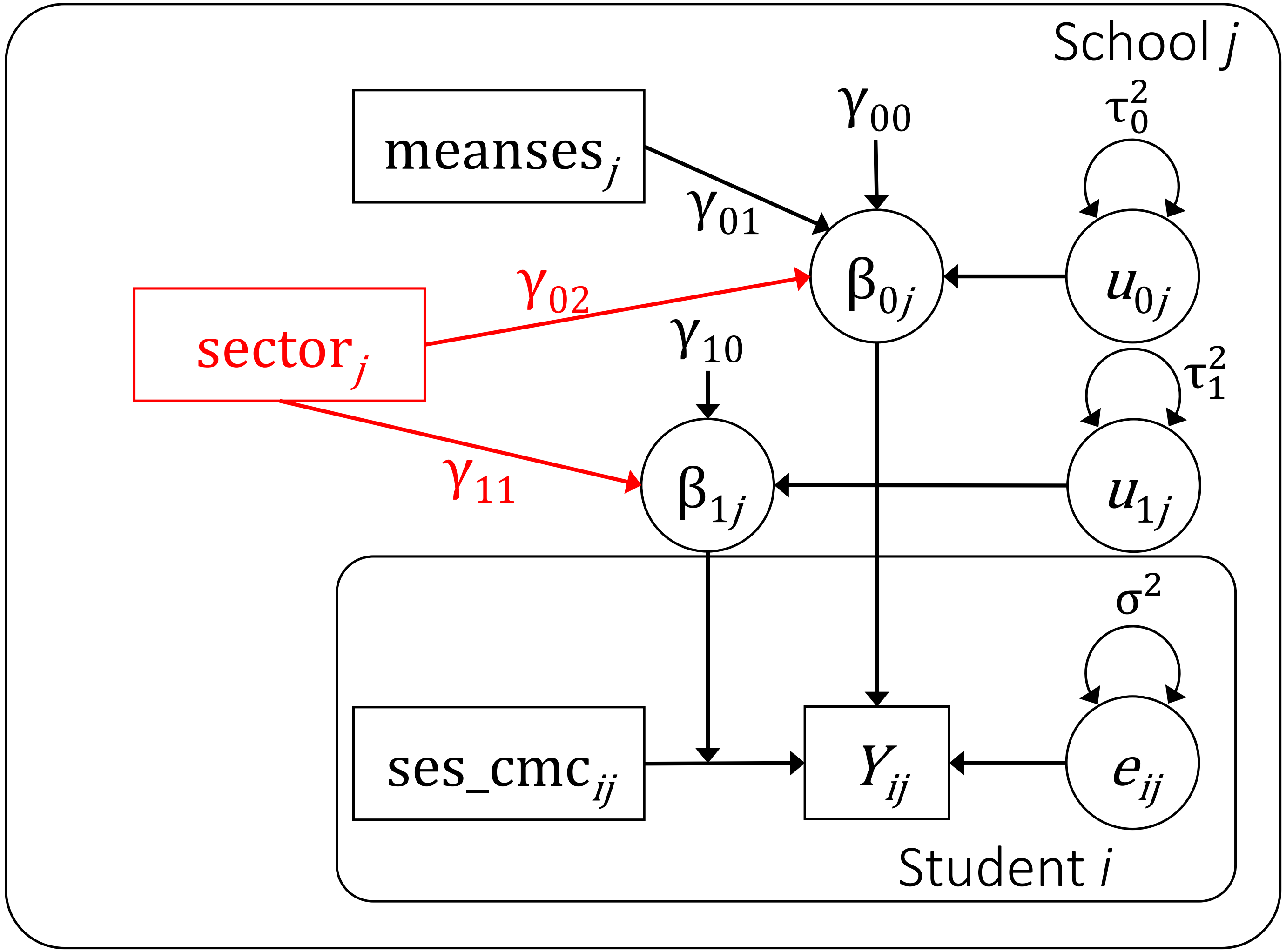
Deviation of intercept for School *j*

Deviation of slope for School *j*

Path Diagram

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{bmatrix}\right)$$

$$e_{ij} \sim N(0, \sigma)$$



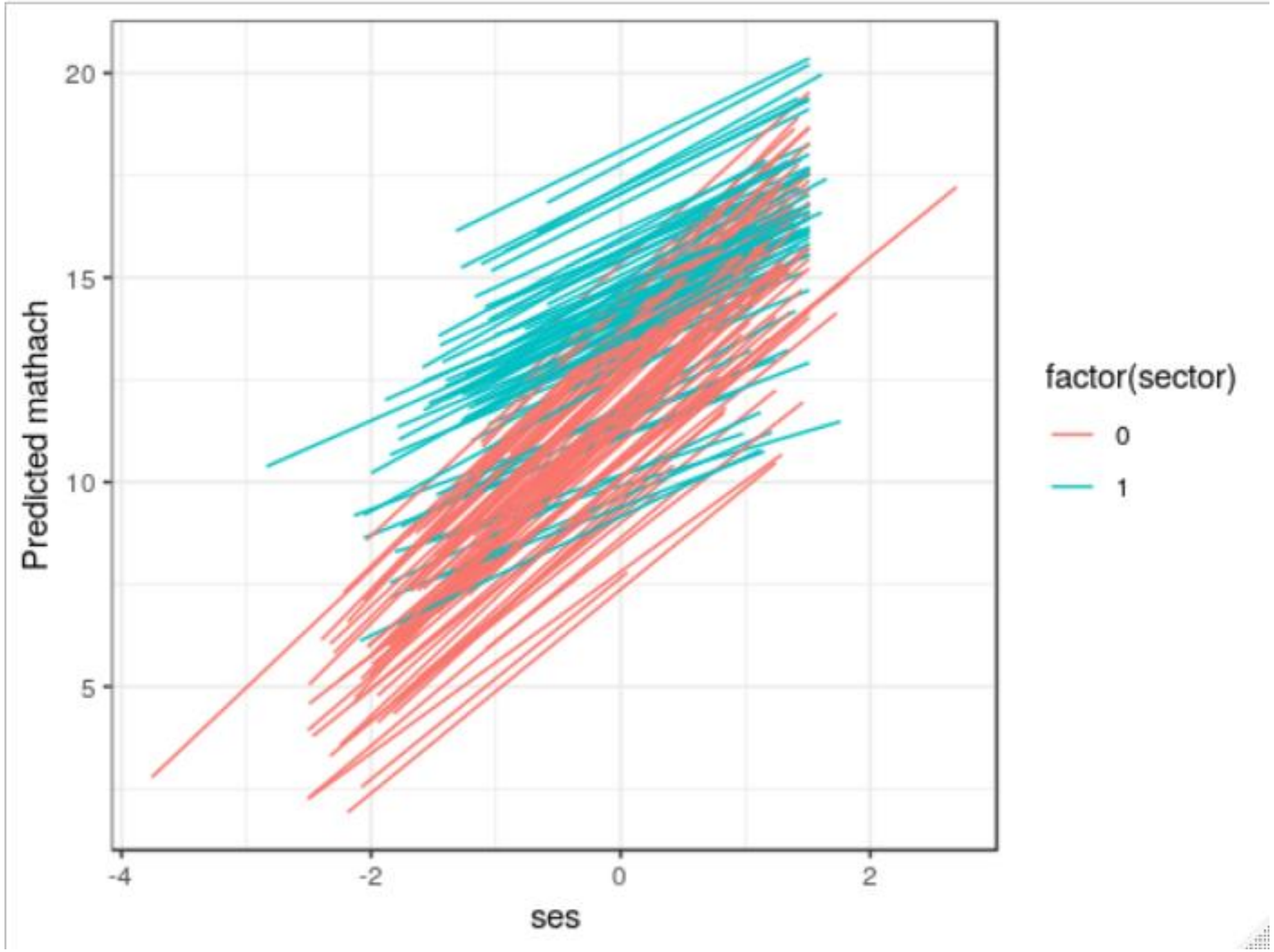
Fixed Effect Estimates

```
Fixed effects:
                      Estimate Std. Error t value
(Intercept)            12.0846     0.1987   60.81
meanses                 5.2450     0.3682   14.24
sectorCatholic          1.2523     0.3062    4.09
ses_cmc                 2.7877     0.1559   17.89
sectorCatholic:ses_cmc -1.3478     0.2348   -5.74
```

Average slope for SES is estimated as **2.79** for Public schools (i.e., sector = 0)

Average slope for SES is estimated as 2.79 – 1.35 = **1.44** for Catholic schools (i.e., sector = 1)

Plot the Interaction



Things to Remember

- A level-1 predictor can have differential relationships with the outcome, depending on the level of analysis
  - Ecological fallacy: assume constant relationship across levels
- Cluster/group-mean centering: decompose a level-1 predictor into its cluster means and deviations from the cluster means
- MLM provides a way to efficiently model variability of regression lines (i.e., intercepts and slopes) across clusters
  - Through the use of random slopes/coefficients
- Cross-level interaction
  = Including a lv-2 predictor in the slope equation